Kinetics of a Verhulst-type system with nonlinearly coupled noise

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(Received 24 June 1996)

The stochastic Bernoulli equation with nonlinearly coupled dichotomous noise is exactly solved by direct averaging. The similar system driven by the periodic perturbation with a random phase is also considered. The results concerning the kinetic and stationary properties in both cases are compared. The evolution of the mean value from the initial states located close to the equilibrium state is found to be nonmonotonic. [S1063-651X(96)06011-4]

PACS number(s): 05.40.+j, 02.50.−r, 05.70.Ln

I. INTRODUCTION

The Bernoulli equation (Verhulst-type model)

$$\dot{x}_t = ax - bx^{\mu+1} \tag{1}$$

is an important example of a nonlinear kinetic equation, which can be exactly solved even if its coefficients $a=a(t)$ and $b=b(t)$ are time dependent. The sufficient condition for the existence of the solution

$$x_t = x_0 e^{\alpha(t)}\left[1+\mu x_0^{\mu}\int_0^t ds e^{\mu\alpha(s)}b(s)\right]^{-\nu}, \tag{2}$$

where $\alpha(s)=\int_0^s du\, a(u)$, $x_0=x_{t=0}>0$, and $\nu=1/\mu$, is

$$\mu b(t)>0 . \tag{3}$$

Considering the influence of (external) noise on such a system, we usually assume that the coefficients of (1) are ''deterministically'' constant, but one of them fluctuates around its average value. The case of (so-called) linearly coupled noise $a(t)=a+\xi_t$, $b(t)=b=\text{const}$ was investigated for different types of noise ξ_t: Gaussian white noise [1], white shot noise [2,3], and the dichotomous Markov (DM) process [4]. Several properties (both stationary and time dependent) of such systems were examined. In contrast, the second case (of nonlinearly coupled noise), when $a(t)=a=\text{const}$ and $b(t)=b+\xi_t$, was touched upon only occasionally [2]. The reason seems to be that the condition (3) cannot be satisfied for Gaussian and for many other types of the noise. The system (1) with nonlinearly coupled Gaussian noise is unstable, and, in particular, no stationary state exists. However, (3) can be satisfied if noise is bounded from below. Then, for each realization of the noise, the solution (2) is well defined and we can (at least in principle) perform the averaging, thus obtaining the transient behavior of the mean value $\langle x_t\rangle$ and of the higher moments as well. Taking the limit $t\to\infty$, we may examine stationary solutions. In this paper we follow this way to discuss the properties of the model (1) for two types of noise

$$b(t)=b[1+\sigma I_t], \quad a(t)=a=\text{const}, \tag{4}$$

where $I_t=\pm 1$ is a DM process [4–6] with $\langle I_t I_s\rangle = e^{-2\lambda|t-s|}$ and $I_t=\cos(\omega t+\phi)$ with *random phase* ϕ.

II. TIME-DEPENDENT SOLUTION IN THE CASE OF DICHOTOMOUS NOISE

Writing (2) as a Laplace-type integral

$$x_t = e^{at}\int_0^\infty ds \frac{s^{\nu-1}}{\Gamma(\nu)} e^{-s\Lambda}\exp\left[-s\mu b\sigma\int_0^t dz e^{\mu a z} I_z\right], \tag{5}$$

where $\Lambda = x_0^{-\mu}+b(e^{\mu a t}-1)/a$, we notice that in order to perform the averaging one needs an expression for

$$\Phi(t)\equiv\left\langle \exp\left[\beta\int_0^t ds e^{\alpha s} I_s\right]\right\rangle . \tag{6}$$

Introducing

$$\Psi(t)\equiv\left\langle I_t \exp\left[\beta\int_0^t ds e^{\alpha s} I_s\right]\right\rangle ,$$

we have

$$\dot{\Phi}=\beta e^{\alpha t}\Psi, \quad \dot{\Psi}=-2\lambda\Psi+\beta e^{\alpha t}\Phi,$$

where the equation for $\dot{\Psi}$ results from Shapiro-Loginov formula [7]. Introducing the new variable $z=-\beta e^{\alpha t}/\alpha$, we obtain

$$\Phi'(z)=\Psi(z), \quad \Psi'(z)+\frac{2\lambda}{\alpha z}\Psi(z)=\Phi(z),$$

so $\Phi(z)$ obeys

$$\Phi''+\frac{2\lambda}{\alpha z}\Phi'-\Phi=0 . \tag{7}$$

The solution of (7) satisfying the initial conditions $\Phi(z=z_0=-\beta/\alpha)=1$ and $\Phi'(z_0)=0$ is given by modified Bessel functions [8]

$$\Phi(z)=\frac{\pi z_0^{\delta+1}z^{-\delta}}{2\sin\pi\delta}[I_{\delta+1}(z_0)I_{-\delta}(z)-I_{-\delta-1}(z_0)I_{\delta}(z)], \tag{8}$$

where

$$\delta=\frac{\lambda}{\mu a}-\frac{1}{2}.$$

Equations (5), (6), and (8) and the integration formula [8]

$$2^{u+v}A^{-u}B^{-v}P^{q}\Gamma(v+1)\int_0^\infty I_u(At)I_v(Bt)e^{-Pt}t^{w-1}dt$$

$$=\sum_{i=0}^{\infty}\frac{\Gamma(q+2i)(A^2/4P^2)^i}{i!\Gamma(u+i+1)}{}_2F_1\left(-i,-u-i;v+1;\frac{B^2}{A^2}\right),$$

where $q\equiv u+w+v>0$ and $P>|A|+|B|$, allow one to calculate the transient mean value

$$\langle x_t\rangle=Q^{\nu}\sum_{i=0}^{\infty}\frac{\{\nu\}_{2i}(gQ)^{2i}}{i!\{1+\delta\}_i}{}_2F_1(-i,-\delta-i;-\delta;\xi^2)$$
$$+\frac{Q^{\nu}\xi^{2\delta+2}}{\delta(\delta+1)}\sum_{i=0}^{\infty}\frac{\{\nu\}_{2i+2}(gQ)^{2i+2}}{i!\{1-\delta\}_i}$$
$$\times{}_2F_1(-i,\delta-i;2+\delta;\xi^2), \tag{9a}$$

where $\xi=e^{-\mu at}$, $Q=(\xi\Lambda)^{-1}$, $g=b\sigma/2a$, and $\{c\}_j\equiv c(c+1)\cdots(c+j-1)=\Gamma(c+j)/\Gamma(c)$. Using Kummer's relations for hypergeometric functions [8], Eq. (9a) may be written as

$$\langle x_t\rangle=Q^{\nu}\sum_{i=0}^{\infty}\frac{\{\nu\}_{2i}}{(2i)!}(gQ\zeta)^{2i}{}_2F_1(i,-\delta+i;2i+1;\zeta), \tag{9b}$$

where $\zeta=1-\xi^2$.

Note that Eq. (1) has the property that its general form is conserved if the ''new variable'' $y_t=x_t^{\omega}$ is introduced. The only change is that coefficients a, b, and $\nu\equiv1/\mu$ and the initial value x_0 are replaced by ωa, ωb, $\omega\nu$, and x_0^{ω}, respectively. Thus Eqs. (9) describe not only the evolution of a mean value, but the evolution of higher moments. The ''composite parameters'' such as δ, g, Q, Λ, and ξ or ζ, are invariant under the above-mentioned transformation, so in order to obtain the expression for $\langle x_t^{\omega}\rangle$ we should only replace ν by $\omega\nu$ in (9). Therefore, Eq. (9) gives a complete (one-point) characterization of a stochastic process x_t.

A. Some properties of the solution (9)

In a number of special cases the series (9) may be summed up. First observe that for $t=0$ we have $Q=x_0^{\mu}$ and $\zeta=0$, so we recover correctly the initial value $\langle x_0\rangle=x_0$. In a second limit $t\to\infty$ we have $Q=a/b$, $\zeta=1$,

$${}_2F_1(i,-\delta+i;2i+1;1)=\frac{(2i)!}{i!\{1+\delta\}_i},$$

and thus

$$\langle x\rangle_{\rm st}=x_{\rm st}\sum_{i=0}^{\infty}\frac{\{\nu\}_{2i}}{i!\{1+\delta\}_i}(\sigma/2)^{2i}$$
$$=x_{\rm st}\,{}_2F_1(\nu/2,\nu/2+1/2;1+\delta;\sigma^2), \tag{10}$$

where $x_{\rm st}=(a/b)^{\nu}$ is a ''deterministic'' stationary state. To obtain the second equality in (10) we have used the relation $\{\nu\}_{2i}=2^{2i}\{\nu/2\}_i\{\nu/2+1/2\}_i$. The stationary mean value exists if $\sigma^2<1$, i.e., when the condition (3) is satisfied. In the case of $\sigma^2>1$ the mean value grows infinitely for finite times.

Let $\delta=-1/2$. Then, the right-hand side (rhs) of Eq. (10) takes a simple form

$$\langle x\rangle_{\rm st}=x_{\rm st}[(1+\sigma)^{-\nu}+(1-\sigma)^{-\nu}]/2$$
$$=\frac{1}{2}\left(\frac{a}{b(1+\sigma)}\right)^{\nu}+\frac{1}{2}\left(\frac{a}{b(1-\sigma)}\right)^{\nu}.$$

This result may be easily understood. The condition $\delta=-1/2$ means that $\lambda=0$, so there is no switching between the two states of the DM process during the evolution. For all t, $I_t=I_0=+1$ or -1 with the equal probability 1/2. Thus the $\langle x\rangle_{\rm st}$ is simply an arithmetical average of two stationary values obtained for deterministic Verhulst systems with parameters $b(1+\sigma)$ and $b(1-\sigma)$, respectively. The same relation should be true for all t. To verify this we insert the identity [9]

$${}_2F_1(i,1/2+i;2i+1;\zeta)=[(1+\sqrt{1-\zeta})/2]^{-2i}$$
$$=2^{2i}(1+\xi)^{-2i}$$

into Eq. (9b), thus obtaining

$$\langle x_t\rangle=x(t;b+\sigma b)/2+x(t;b-\sigma b)/2\equiv X(\sigma), \tag{11}$$

where

$$x(t,b)=[x_0^{-\mu}e^{-\mu at}+b(1-e^{-\mu at})/a]^{-1/\mu} \tag{12}$$

is the solution of (1) for constant a and b parameters.

The less trivial example is that for $\delta=1/2$, which is when $\lambda=\mu a$. Using the identity [9]

$${}_2F_1(i,i-1/2;2i+1;\zeta)=2^{2i}(1+\xi)^{-2i}\left[\xi+\frac{1-\xi}{2i+1}\right],$$

the rhs of (9b) may be summed up to the form

$$\langle x_t\rangle=\xi X(\sigma)+\frac{1-\xi}{\sigma}\int_0^{\sigma}d\sigma' X(\sigma')$$
$$=e^{-\mu at}X(\sigma)+\frac{a}{2b\sigma}$$
$$\times\begin{cases}\dfrac{1}{1-\nu}[x^{1-\mu}(t;b+\sigma b)-x^{1-\mu}(t;b-\sigma b)] & \text{if } \nu\neq1\\ \ln[x(t;b-\sigma b)/x(t;b+\sigma b)] & \text{if } \nu=1\end{cases} \tag{13}$$

[we use the notation of Eqs. (11) and (12)]. Taking the limit $t\to\infty$, we obtain

$$\langle x\rangle_{\rm st}=x_{\rm st}\frac{(1+\sigma)^{1-\nu}-(1-\sigma)^{1-\nu}}{2(1-\nu)\sigma}\quad(\nu\neq1)$$
$$=x_{\rm st}\frac{1}{2\sigma}\ln\frac{1+\sigma}{1-\sigma}\quad(\nu=1),$$

which are the particular results of (10) [9].

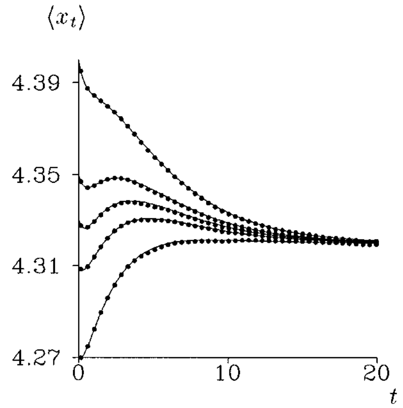

FIG. 1. Comparison of analytical results and digital simulation. Plots of $\langle x_t \rangle$ vs t (rescaled, dimensionless time; $\lambda=1$) [Eq. (9)] for $a=0.08$, $b=0.001$, $\mu=3$, $\sigma=0.8$, and $x_0=4.27,4.31,4.33,4.35,4.40$, respectively. Each mark represents the arithmetical average over $N=10^7$ values obtained for different sample realizations.

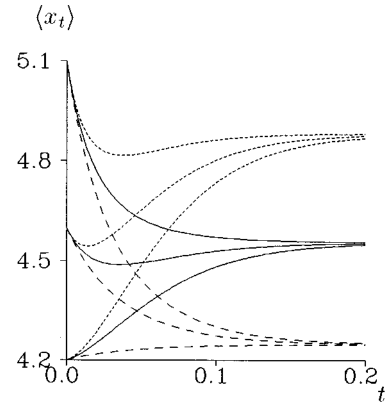

FIG. 3. Plots of $\langle x_t \rangle$ vs t (rescaled, dimensionless time; $\lambda=\omega=1$) for different initial states x_0. The solid lines correspond to the kinetics driven by periodic perturbation [Eq. (14)], the dotted lines correspond to the case of dichotomous noise [Eq. (9)], and the dashed lines correspond to the deterministic system [Eq. (12)], respectively. $\mu=3$, $a=8$, $b=0.1$, $\sigma=0.7$, and $x_0=4.2,4.6,5.1$.

B. Transient behavior: Numerical results and simulation

The explicit time dependence of $\langle x_t \rangle$ can be easily computed by truncating the infinite series (9a) or (9b) after approaching the required level of accuracy. For small and ''intermediate'' t (when $\zeta \lesssim 0.8$–0.9) it is better to use Eq. (9b), whereas for large t the representation (9a) is the more appropriate one [10]. When the initial value x_0 is far from the stationary one $\langle x \rangle_{\mathrm{st}}$ the relaxation of $\langle x_t \rangle$ turns out, like for the deterministic solution (12), to be monotonic. A more interesting effect is observed if the initial state is close to the equilibrium value. Such a situation is presented in Fig. 1, namely, we observe that the relaxation is nonmonotonic. A local minimum, followed usually by a local maximum, appears on the plot $\langle x_t \rangle$ vs t. The appearance of the local minimum in some cases may be easily explained as follows. From Eq. (10) we know that the equilibrium state of a noisy system is located higher than the deterministic one ($\langle x \rangle_{\mathrm{st}} > x_{\mathrm{st}}$). On the other hand, $\langle \dot{x}_{t=0^+} \rangle = a x_0 - b x_0^{\mu+1}$, so in the case of $x_0 > x_{\mathrm{st}}$ the first derivative of $\langle x_t \rangle$ at $t=0$ is negative. Thus, at the beginning of evolution the system is forced down, which may be against the global trend $x_0 \rightarrow \langle x \rangle_{\mathrm{st}}$.

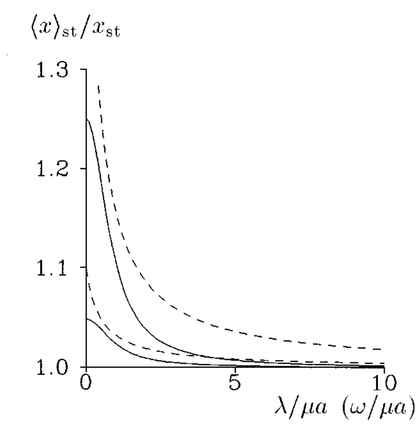

FIG. 2. Plots $\langle x \rangle_{\mathrm{st}}/x_{\mathrm{st}}$ vs $\lambda/\mu a$ in the case of a DM process (dashed lines) or vs $\omega/\mu a$ in the case of a periodic perturbation (solid lines). $\mu=1$ and $\sigma=0.3$ (lower curves), or $\sigma=0.6$ (upper curves).

In order to confirm the analytical results we have done the digital averaging of (2). For each realization of the dichotomous Markov process I_t, $t \in (0,T)$ [where T is an arbitrarily given ending time], the value of x_t is given elementarily by

$$x_t(\{t_i\}) = x_0 e^{at} \left\{ 1 + b x_0^{\mu} \sum_{i=0}^{i(t)} [1+(-1)^i \sigma I_0] \times (e^{\mu a t_{i+1}} - e^{\mu a t_i})/a \right\}^{-1/\mu},$$

where $\{t_i\}$ is a sequence of random points on the time interval $(0,T)$ governed by Poisson process with parameter λ; $t_0=0$ and $t_{i(t)+1}=t \leqslant T$. $I_0=+1$ or -1 with the probability 1/2.

The $\langle x_t \rangle$ is then calculated as the arithmetical average of several thousand values, obtained for different sample realizations values. A comparison of analytical results and digital simulation is given in Fig. 1.

III. PERIODIC PERTURBATION WITH RANDOM PHASE

Now consider the case when the nonlinearly coupled perturbation is given by $I_t = \cos(\omega t + \phi)$, where ϕ is a random phase uniformly distributed on $(0,2\pi)$. This process and the previously considered DM process have some common properties: both are stationary, their values are bounded in between -1 and $+1$, and both depend on a single parameter of frequency units. Therefore a natural suggestion follows to compare the properties of Verhulst-type systems driven by both types of noise.

The mean value of x_t is simply given by an integral $\langle x_t \rangle = (2\pi)^{-1} \int_0^{2\pi} d\phi \, x_t(\phi)$, where $x_t(\phi)$ is a deterministic

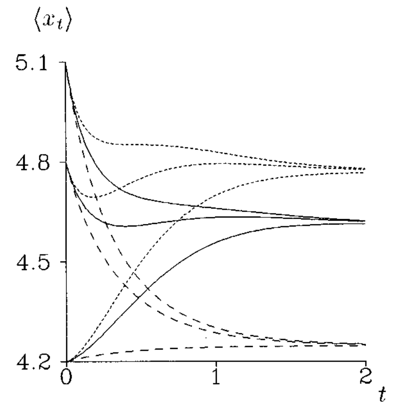

FIG. 4. Same as Fig. 3, but for $\mu=3$, $a=0.8$, $b=0.01$, $\sigma=0.8$, and $x_0=4.2,4.8,5.1$.

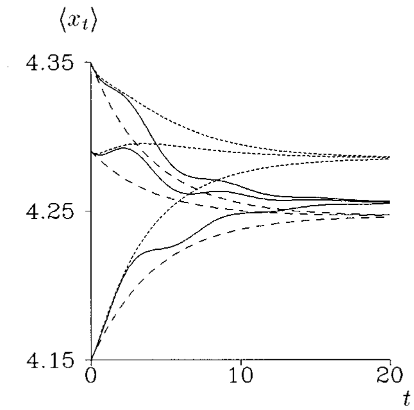

FIG. 5. Same as Fig. 3, but for $\mu=3$, $a=0.08$, $b=0.001$, $\sigma=0.6$, and $x_0=4.15,4.29,4.35$.

solution obtained for fixed but otherwise arbitrary ϕ. The identity

$$\int_0^{2\pi}\frac{d\phi}{2\pi}[1+\beta\cos\phi-\gamma\sin\phi]^{-\nu}={}_2F_1\left(\frac{\nu}{2};\frac{\nu+1}{2};1;\beta^2+\gamma^2\right)$$

allows one to obtain

$$\langle x_t\rangle=Q^{\nu}{}_2F_1\left(\frac{\nu}{2},\frac{\nu+1}{2};1;\frac{(b\sigma Q)^2}{a^2}\frac{[1-2\xi\cos\omega t+\xi^2]}{1+(\omega/\mu a)^2}\right), \tag{14}$$

where the notation of Sec. II is being used. Thus, for the stationary state we have

$$\langle x\rangle_{\rm st}=x_{\rm st}\,{}_2F_1\left(\frac{\nu}{2},\frac{\nu+1}{2};1;\frac{\sigma^2}{1+(\omega/\mu a)^2}\right). \tag{15}$$

If $\omega\neq 0$ the convergence of (15) is possible even for some values of $\sigma^2>1$. However, in order to keep the solution (14) finite for any time and for an arbitrary initial state x_0, the condition $\sigma^2<1$ is again required. In Fig. 2 we plot the rhs of (10) and (15) as a function of dimensionless frequency parameter $\lambda/\mu a$ $(=\omega/\mu a)$. We see that dichotomous noise shifts the $\langle x\rangle_{\rm st}$ to higher values than the periodic noise does. In both cases the effect is strong if the external (noisy) time scale $T_{\rm ext}\sim\lambda^{-1}=\omega^{-1}$ is greater than or of the order of the system's time scale $T_{\rm int}\sim a^{-1}$. In the case of $T_{\rm ext}\ll T_{\rm int}$ the stationary value changes are relatively small.

Figures 3–5 show the relaxation of the deterministic system $(\sigma=0)$ and of two related stochastic systems driven by the DM process or periodic perturbation, respectively. We see that for any initial state x_0 and for an arbitrary $t>0$ the deterministic state (12) is located below the mean value (9) obtained for the dichotomous noise case and in between lies the corresponding mean value of (14) (the case of periodic perturbation). The same picture was observed for different values of the system's parameters and seems to be a general rule. In Fig. 5 we may observe a periodic modulation on the plot of (14).

IV. FINAL REMARKS

We have solved exactly the stochastic Verhulst model with nonlinearly coupled dichotomous noise. The rather simple formula (9b) describes the evolution of the mean value (in fact, it gives the full one-point characterization of x_t, namely, its transient Mellin function) for any deterministic initial state x_0. The evolution of other states may be obtained by a simple integration of (9) over the initial probability density distribution $P(x_0)$. It was found that the relaxation may be nonmonotonic; compare Figs. 1 and 3–5. The analytical results have been confirmed by computer simulation (Fig. 1).

We have compared (Figs. 3–5) the kinetics of the deterministic case (12) with the kinetics driven by periodic perturbation with a randomly distributed phase [Eq. (14)] and with the kinetics driven by dichotomous noise [Eq. (9)]. The mean value of x_t was the greatest in the case of the dichotomous process, the lowest for the deterministic system, and in between for the system driven by periodic noise.

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[9] This results from a so-called quadratic transformation of the hypergeometric functions; see Ref. [8].

[10] When $\zeta \rightarrow 1$ the hypergeometric series of (9b) become slowly convergent and simultaneously the parameters of expansions in (9a) and (9b) $[(gQ)^2$ and $(gQ\zeta)^2$, respectively] become almost equal.